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A POWER SERIES SOLUTION FOR THE NON-LINEAR VIBRATION OF BEAMS

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A power series solution is presented for the non-linear free vibration of beams with restrained ends. The analysis is based on transforming the time variable into an oscillating time which allows the motion of the beam, assumed to be periodic, to be expressed as a double power series that is convergent for all time. A recurrence relation is used to determine the series coefficients, with the initial movement satisfying the boundary conditions as its basis. Results are obtained for simply supported and clamped beams and compared with available solutions.

1. INTRODUCTION

The study of the free vibration of geometrically non-linear beams involves obtaining solutions to governing non-linear partial differential equations for which exact analysis is not available. Two approximate methods have traditionally been used in the analysis of such problems. In the first method [1, 2], the beam is assumed to vibrate in its linear mode shape, which serves to reduce the equation of motion of a modal Duffing equation, for which an exact solution exists in the form of an elliptic integral. In the second method [3, 4] one assumes a harmonic time dependence which reduces the problem to that of solving a differential equation in the space variable. More recently, a method based on the concept of invariant manifolds [5] was used to generate normal modes of vibration for weakly non-linear systems.

In the past, power series expansions of dynamical motions were not possible because the infinite extent of the time variable gives rise to secular terms. However, by transforming the time variable into an oscillating time, the governing differential equation becomes, under certain conditions relating to the invertibility of the transformation, well-conditioned for a solution by the power series method. In this paper, the problem of free vibration of non-linear beams with restrained ends was chosen for power series analysis because such conditions are satisfied by the beam motion. To that extent, the present approach is not to be regarded as generally applicable.

2. EQUATION OF MOTION

For moderately large amplitude vibrations, the equation of motion of beams with their ends restained in the horizontal x-direction is given by

$$\ddot{W} + \alpha W'''' - \beta\left(\int_0^1 W'^2\,\mathrm{d}\zeta\right)W'' = 0 \tag{1}$$

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where the overdot and prime denote differentiation with respect to the time t and the space variable $\zeta = x/L$, respectively. The beam motion is described by the transverse displacement W and the constants $\alpha = EI/mL^4$ and $\beta = EA/2mL^4$ in which E, I, m, A and L are the modulus of elasticity, the moment of inertia of the cross-section, the mass per unit length, the cross-sectional area and the length of the beam, respectively. It is assumed that the beam is linearly elastic and that horizontal inertia forces are negligible in comparison with transverse inertia forces. In addition, shear deformation and rotary inertia are ignored.

The theory of power series expansion [6] places as a condition for convergence the requirement that the independent variables involved in the expansion be of finite extent. In vibration analysis, the time variable t has an infinite domain and, consequently, power series solutions may be obtained only within a small interval of time. To facilitate the use of the power series method in capturing the periodic motion of dynamical systems, it is proposed to transform the time variable t into an oscillating time τ as follows:

$$\tau = h(t) = \sin \omega t. \tag{2}$$

This transforms the infinite time domain $0 \leqslant t \leqslant \infty$ into a finite time scale $-1 \leqslant \tau \leqslant 1$ within which τ oscillates harmonically at a frequency ω to be determined. In addition, equation (2) satisfies the requirement that $\tau = 0$ at the initial instant $t = 0$. This requirement is particularly important when the initial conditions are considered. It will be shown that, under certain conditions, the time transformation (2) is invertible and the beam displacement can therefore be expressed in two different but equivalent functional forms $w(\zeta, \tau) = W(\zeta, h^{-1}(\tau))$. When equation (2) is used to transform equation (1) into the (ζ, τ) co-ordinates, only the first term is affected and the equation of motion becomes

$$\omega^2(1 - \tau^2)w_{\tau\tau} - \omega^2 \tau w_\tau + \alpha w'''' - \beta\left(\int_0^1 w'^2 \, \mathrm{d}\zeta\right) w'' = 0, \tag{3}$$

in which the subscript τ denotes differentiation with respect to τ. The transformation is seen to change the character of the equation into a non-autonomous one wherein the time variable appears explicitly. Also, the frequency ω of the oscillating time is an auxillary parameter which influences the solution. Equation (3) is a non-linear partial differential equation which is to be solved subject to four boundary conditions and two initial conditions. The boundary conditions, two at each end of the beam, remain unchanged by the transformation. For the initial conditions, only the velocity in the new co-ordinates is affected by ω,

$$w(\zeta, 0) = W_0(\zeta, 0), \qquad w_\tau(\zeta, 0) = \dot{W}_0(\zeta, 0)/\omega, \tag{4}$$

where $W_0(\zeta, 0)$ and $\dot{W}_0(\zeta, 0)$ are the initial displacement and velocity, respectively, in the original co-ordinates.

3. SOLUTION

According to differential equation theory [6], equation (3) has an ordinary point at $\tau = 0$, two regular singular points at $\tau = \pm 1$ and an ordinary point at $\zeta = 0$. Thus, in view of the finite ranges of the independent variables involved, it is possible to expand the beam motion as a double power series about the ordinary points $\zeta = 0$ and $\tau = 0$ as

$$w(\zeta, \tau) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \zeta^{m-1} \tau^{n-1}, \tag{5}$$

in which a_{nm} are constant coefficients to be determined. By substituting equation (5) into equation (3) and making appropriate changes of indices so that all terms have the same power, the governing equation can be reduced to

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} \zeta^{m-1} \tau^{n-1}, \tag{6}$$

where $c_{nm} = \omega^2[n(n+1)a_{n+2,m} - (n-1)(n-2)a_{nm}] - \omega^2(n-1)a_{nm} + \alpha m(m+1)(m+2)(m+3)a_{n,m+4} - \beta b_{nm}$. The coefficients b_{nm} of the non-linear term result from the multiplication of two power series: one in terms of τ only representing the integral $\int_0^1 w'^2 \, \mathrm{d}\zeta$ and the other a double series for the derivative w''. Consequently, such coefficients consist of terms involving products of the constants a_{nm}. In fact, the value of b_{nm} can be computed once $a_{nm}, a_{n-1,m}, a_{n-2,m}, \ldots, a_{1m}$ are known. Now equation (6) is satisfied for all possible values of ζ and τ only if all the coefficients c_{nm} vanish. This condition gives the recurrence relation

$$a_{n+2,m} = \frac{\omega^2(n-1)^2 a_{nm} - \alpha m(m+1)(m+2)(m+3)a_{n,m+4} + \beta b_{nm}}{\omega^2 n(n+1)},$$
$$n, m = 1, 2, 3, \cdots, \tag{7}$$

between the series coefficients. The basis for this relation is the initial displacement and velocity equations (4) which determine the elements of the first and second rows of the coefficient matrix: a_{1m} and a_{2m}, respectively. The elements of the remaining rows are then determined recursively from equation (7) and their values are, in general, dependent on ω. The introduction of the oscillating time frequency as an additional unknown parameter in the equation calls for an auxiliary condition for its determination. Such a condition is provided by Rayleigh's energy principle which stipulates that, for a conservative system, the maximum kinetic and strain energies are equal. For beams with restrained ends, the kinetic and strain energies are, respectively, given by

$$T = \frac{mL}{2} \int_0^1 \dot{w}^2 \, \mathrm{d}\zeta = \omega^2(1 - \tau^2) \frac{mL}{2} \int_0^1 w_\tau^2 \, \mathrm{d}\zeta$$

and

$$U = \frac{\alpha mL}{2} \int_0^1 (w'')^2 \, \mathrm{d}\zeta + \frac{\beta mL}{4} \left(\int_0^1 w'^2 \, \mathrm{d}\zeta \right)^2. \tag{8, 9}$$

The series coefficients a_{nm} depend only on ω and consequently these energy expressions are functions of ω and τ. Suppose that the motion starts from rest at $t = \tau = 0$ with a prescribed displacement that satisfies the boundary conditions. At this instant, the strain energy is at a maximum and can be computed from equation (9). Since the beam vibration at moderately large amplitude is periodic with frequency Ω say, and the beam motion is described in terms of τ, the motion must repeat itself every time τ is zero. This corresponds to one half-cycle of ω and, consequently, the frequency of the oscillating time equals one half the vibration frequency:

$$\omega = \Omega/2. \tag{10}$$

The introduction of zero initial velocity into the recurrence equation (7) results in the vanishing of all the even rows in the coefficient matrix. Consequently, the beam motion, as represented by equation (5), is an even function of τ, so that, as time t progresses from zero, each half-cycle of the oscillating time captures repeatedly the full cycle of beam vibration. Furthermore, the symmetry of the oscillating time requires that both halves of the vibration cycle be identical. It follows that under the conditions of periodicity, zero initial velocity and symmetry of the vibration cycle, the time transformation given in equation (2), which converts time into a non-monotonic function, becomes invertible.

The equilibrium position corresponding to maximum velocity is reached at angular positions $\Omega t = \pi/2,\ 3\pi/2,\ 5\pi/2, \ldots$ for which $\tau = \pm 1/\sqrt{2}$. At these instants, the kinetic energy is at a maximum and may be determined from equation (8). The oscillating time frequency is obtained from a frequency search for a value of ω that satisfies Rayleigh's energy principle. The vibration frequency Ω can, as a result, be determined to the desired degree of accuracy. Now, since the natural motion of the beam at moderately large amplitude is at present unknown, the initial displacement may arbitrarily be chosen as any one of the free vibration modes of the linearized beam with the initial velocity taken to be zero: i.e.,

$$w(\zeta, 0) = \lambda g_n(\zeta), \qquad w_\tau(\zeta, 0) = 0, \tag{11}$$

where $g_n(\zeta)$ is the nth normal linear mode of vibration, $g_n(\bar{\zeta}) = 1$ for a freely chosen $\zeta = \bar{\zeta}$ and λ denotes the amplitude of vibration at the point $\zeta = \bar{\zeta}$. Also, it is common practice to express the order of non-linearity in terms of a vibration amplitude ratio λ/k, where $k = \sqrt{I/A} = \sqrt{\alpha/2\beta}$ is the radius of gyration of the beam cross-section. The linearized beam corresponds to $\beta = 0$ with the vibration amplitude ratio assuming small values ($\lambda/k \ll 1$).

4. NUMERICAL EXAMPLES

Power series solutions of equation (3) were computed for simply supported and clamped beams. In each case, a convergent solution was obtained. The linear modes of vibration were used as initial displacements which could be expanded as power series, so that values for all the elements of the first row in the coefficient matrix may be assigned. All the elements of the second row representing the initial velocity vanished since the initial velocity was taken zero.

For the simply supported beam

$$g_n(\zeta) = \sin n\pi\zeta = (n\pi)\zeta - \frac{(n\pi)^3}{3!}\zeta^3 + \frac{(n\pi)^5}{5!}\zeta^5 - \ldots, \tag{12}$$

so that $a_{11} = 0$, $a_{12} = n\pi\lambda$, $a_{13} = 0$, $a_{14} = -\{(n\pi)^3/3!\}\lambda$, etc. The boundary conditions $w(0, \tau) = 0$, $w''(0, \tau) = 0$ at $\zeta = 0$ require that the first and third columns of the coefficient matrix be zero if the ensuing motion is to satisfy these conditions. Two additional requirements are placed on the beam motion from the boundary conditions $w(1, \tau) = 0$, $w''(1, \tau) = 0$ at $\zeta = 1$. These are

$$\sum_{m=1}^{\infty} a_{nm} = 0, \qquad \sum_{m=1}^{\infty} m(m+1)a_{n,m+2} = 0, \qquad n = 1, 2, 3, \ldots. \tag{13}$$

TABLE 1

Vibration frequency ratio Ω/Ω_L for a simply supported beam ($\alpha = 2$, $\beta = 1$).

	First Mode		Present		
λ/k	Lewandowski [7]	Krieger [1]	First mode	Second mode	Third mode
1	1·0897	1·0892	1·0892	1·0895	1·0894
2	1·3299	1·3178	1·3178	1·3179	1·3179
3	1·6394	1·6257	1·6257	1·6228	1·6260

These requirements are uniquely satisfied by the natural "non-linear" modes of vibration of the beam and may therefore be used as a criterion for qualifying the initial displacement as a natural mode. In Table 1 is given the vibration frequency ratio Ω/Ω_L of the first three modes for different values of λ/k, and the results for the first mode are compared with available results. The frequency Ω_L relates to the linear beam. In each case, the coefficient matrix size was 40×40. The results for the first mode are identical to those of Krieger [1] who used a linear mode assumption, with those of Lewandowski [7] using the Ritz method providing an overestimate. The frequency ratios of the second and third modes are seen to be practically the same as those of the first mode, which suggests that the frequency ratio of the simply supported beam is independent of the mode number. This result is in agreement with the findings of Krieger [1]. In Table 2 are given the first 11×6 non-zero coefficients of the series matrix for the first mode with $\lambda/k = 2$.

For beams clamped at both ends,

$$g_n(\zeta) = [\sin \gamma_n\zeta - \sinh \gamma_n\zeta - \eta_n(\cos \gamma_n\zeta - \cosh \gamma_n\zeta)]/\psi_n, \tag{14}$$

where

$$\eta_n = (\sin \gamma_n - \sinh \gamma_n)/(\cos \gamma_n - \cosh \gamma_n),$$

$$\psi_n = (\sin \gamma_n\bar{\zeta}_n - \sinh \gamma_n\bar{\zeta}_n) - \eta_n(\cos \gamma_n\bar{\zeta}_n - \cosh \gamma_n\bar{\zeta}_n),$$

and γ_n are the positive roots of the transcendental equation $\cos \gamma \cosh \gamma = 1$.

Similarly, this mode shape may be expanded into a single power series, the coefficients of which are assigned to the elements of the first row of the series matrix. In Table 3 the frequency ratios for a clamped beam are compared with those of the Ritz method [7] for

TABLE 2

The non-zero series coefficients for first mode of a simply supported beam ($\alpha = 2$, $\beta = 1$)

n	$m = 2$	$m = 4$	$m = 6$	$m = 8$	$m = 10$	$m = 12$
1	6·2857	−10·3479	5·1106	−1·2019	0·1649	−0·0148
3	−14·5046	23·8784	−11·7930	2·7735	−0·3805	0·0342
5	6·3230	−10·4093	5·1409	−1·2090	0·1659	−0·0149
7	−6·2983	10·3687	−5·1209	1·2043	−0·1652	0·0148
9	3·7806	−6·2238	3·0738	−0·7229	0·0992	−0·0089
11	−3·0152	4·9639	−2·4515	0·5766	−0·0791	0·0071
13	2·0356	−3·3511	1·6550	−0·3892	0·0534	−0·0048
15	−1·5110	2·4874	−1·2285	0·2899	−0·0396	0·0036
17	1·0624	−1·7489	0·8638	−0·2031	0·0279	−0·0025
19	−0·7697	1·2672	−0·6258	0·1472	−0·0202	0·0018
21	0·5489	−0·9036	0·4463	−0·1050	0·0144	−0·0013

TABLE 3

Vibration frequency ratio Ω/Ω_L for a clamped beam ($\alpha = 2,\ \beta = 1$).

	First mode		Second mode		Third mode	
λ/k	Lewandowski	Present	Lewandowski	Present	Lewandowski	Present
1	1·0222	1·0167	1·0486	1·0143	1·0605	1·0034
2	1·0858	1·0739	1·1794	1·1133	1·2169	1·0840
3	1·1833	1·1613	1·3635	1·2661	1·4312	1·2325

different values of the amplitude ratio. Here, significant differences in frequency ratio are seen to exist, which may be attributed to two factors. First, the Ritz method, like any approximate technique, provides an overestimate of the frequency. Second, an attempt was made to minimize these differences by increasing the number of polynomial terms. This, however, did not produce the desired effect, which suggests that the frequencies predicted by the present method are significantly influenced by the prescribed initial displacement. The first 11×6 non-zero series coefficients for the first mode are shown in Table 4 for $\lambda/k = 2$. The boundary conditions at $\zeta = 0$ for a clamped beam require that the first and second columns of the coefficient matrix be zero, a condition which is clearly violated, as depicted in Table 4. The corresponding requirements for the simply supported beam; namely, the vanishing of the first and third columns, were always satisfied. At the boundary $\zeta = 1$, the conditions $w(1, \tau) = 0$, $w'(1, \tau) = 0$ for the clamped beam require that

$$\sum_{m=1}^{\infty} a_{nm} = 0, \qquad \sum_{m=1}^{\infty} m a_{n,m+1} = 0, \qquad n = 1,\ 2,\ 3, \ldots . \tag{15}$$

In Table 5 are shown the summations involved in the boundary conditions at $\zeta = 1$ for the first mode of the two beams ($\lambda/k = 2$). The values for $n = 1$ representing those of the linear mode shapes are not zero because of the truncations involved. Values for $n \geqslant 3$ clearly shown that the boundary conditions at $\zeta = 1$ are satisfied by the simply supported beam and violated by the clamped beam. Consequently, it may be concluded that the natural modes of vibration at large amplitude for simply supported beams are the same as the linear ones. This result is supported by the agreement of the computed frequencies

TABLE 4

The non-zero series coefficients for the first mode of clamped beam ($\lambda/k = 2$)·

n	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
1	0·0000	0·0000	28·2090	−43·7244	0·0000	0·0000
3	3·9142	−1·6547	0·0000	0·0000	0·0818	−0·0253
5	1·9035	−8·8516	−48·9245	75·8338	39·7967	−37·0114
7	−6·7887	2·8698	0·5020	−0·1796	−0·1418	0·0440
9	−2·4386	11·3395	0·05194	−0·8051	−50·9823	47·4141
11	0·0721	−0·0305	0·6431	0·2300	−0·0395	0·0122
13	2·0834	−9·6879	−7·8382	12·1493	43·5570	−40·5084
15	−0·8607	0·3639	0·4991	−0·1785	− 0·0426	0·0132
17	−1·4953	6·9531	8·5106	−13·1916	−31·2610	29·0730
19	1·3777	−0·5824	−0·3578	0·1280	0·0035	−0·0011
21	1·0367	−4·8206	−8·8853	13·7724	21·6575	−20·1417

TABLE 5

The boundary condition satisfaction criterion at $\zeta = 1$ for the first mode ($\lambda/k = 2$)

	Simply supported beam		Clamped beam	
n	$\sum_{m=1} a_{nm}$	$\sum_{m=1} m(m+1)a_{n,m+2}$	$\sum_{m=1} a_{nm}$	$\sum_{m=1} ma_{n,m+1}$
1	−0·0025	0·0249	0·0000	0·0336
3	0·0058	−0·0576	1·9100	8·8060
5	−0·0025	0·0251	−2·4711	−11·6834
7	0·0025	−0·0250	2·1723	10·3044
9	−0·0015	0·0150	−1·3747	−5·6442
11	0·0012	−0·0119	1·0475	5·1886
13	−0·0008	0·0081	−0·8739	−3·6587
15	0·0006	−0·0060	0·6189	3·1822

with those obtained by using the linear mode assumption [1]. For clamped beams, however, the linear modes are different from the non-linear modes.

It is also appropriate to evaluate the stretching force F which is responsible for the non-linear vibration of beams with restrained ends. In Table 6 are given for the first mode, the amplitude of the stretching force ratio

$$\frac{FL^2}{EI} = \frac{\beta}{\alpha}\int_0^1 (w')^2\,\mathrm{d}\zeta \tag{16}$$

of the two beams for different values of vibration amplitude ratio, and the results are compared with those presented in reference [1, 3]. Good agreement is shown for both beams. It is evident from equation (16) that the stretching force is constant over the beam length but is time-dependent. It was found that, for the simply supported beam, the stretching force vibrates with a frequency twice that of the beam vibration frequency. This is in agreement with the results obtained by Krieger [1].

Dowell [8] has shown that, if the non-linear mode of vibration is assumed to be a combination of the linear mode shapes

$$w(x, t) = \sum_{i=1}^{n} q_i(t)w_i(x)$$

the presence of the non-linear stretching force produces coupling among the mode shapes such that the frequency ratios for the second and higher modes are always greater than those for the first mode. Here, it is not possible to verify this finding for the clamped beam in the results because of the influence of the prescribed displacement, or in the formulation

TABLE 6

The stretching force ratio FL^2/EI for beams ($\alpha = 2$, $\beta = 1$)

	Simply supported beam		Clamped beam	
λ/k	Krieger [1]	Present	Sarma *et al*. [3]	Present
1	2·4674	2·4704	2·4381	2·4403
2	9·8696	9·8816	9·7452	9·7612
3	22·2066	22·2334	21·9087	21·9632

because the frequency is intertwined in the coefficients in a complex way. However, this assumption breaks down for the simply supported beam, for which the linear and non-linear modes are identical. It may, therefore, be argued that the non-linear modes of vibration must form an independent set of functions. Under this condition of mode independence no coupling takes place due to the stretching force, which results in frequency ratios being independent of the mode number. One may thus predict that the aforementioned assumption leads to overestimates of the higher mode frequencies.

5. CONCLUSIONS

A power series approach has been presented for the analysis of non-linear vibration of beams with restrained ends. Convergent solutions were obtained upon transforming the time variable into an harmonically oscillating time. The frequency of the oscillating time is obtainable from Rayleigh's energy principle. The basic approach may be applied to systems with quadratic or other non-linearities, provided that the conditions for invertibility of time transformation are satisfied. It may not, for instance, be applied to asymmetric waveforms or to transient motion caused by damping, since these violate the periodicity condition. Results for simply supported and clamped beams showed agreement of the computed vibration frequencies with those available in the literature. A criterion for non-linear modes was introduced which showed that simply supported beams vibrate at large amplitudes with their linear modes. However, this result was not valid for clamped beams.

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